Generalized double extension and descriptions of quadratic Lie superalgebras

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Abstract

A Lie superalgebra endowed with a supersymmetric, even, non-degenerate, invariant bilinear form is called a quadratic Lie superalgebra. In this paper we give an inductive description of quadratic Lie superalgebras in terms of generalized double extensions; more precisely, we prove that every quadratic Lie superalgebra may be constructed by sucessive orthogonal sums and/or generalized double extensions. In particular, we prove that all solvable quadratic Lie superalgebras are obtained by a sequence of generalized double extensions by one-dimensional Lie superalgebras. We also prove that every solvable quadratic Lie superalgebra is isometric to either a T^*-extension of certain Lie superalgebra or to an ideal of codimension one of a T^*-extension.

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Introduction

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ a finite dimensional Lie superalgebra over a commutative algebraically closed field $\mathbb{K}$ of characteristic zero. An invariant scalar product B on $\mathfrak{g}$ is an even, supersymmetric, nondegenerate bilinear form on $\mathfrak{g}$ which is also invariant, this is to say, the equality $B([X,Y],Z) = B(X,[Y,Z])$ holds for every $X, Y, Z \in \mathfrak{g}$. In this case, we say that $(\mathfrak{g}, B)$ is a quadratic (or orthogonal or metrised) Lie superalgebra. The basic classical Lie superalgebras ([12], [19]) and semisimple Lie algebras endowed with the Killing form are, therefore, examples of quadratic Lie superalgebras, but there are many other Lie superalgebras which admit invariant scalar products even when the Killing form is identically zero.

Quadratic Lie algebras have been widely studied since they appear in connection with many problems derived from Geometry, Physics and other disciplines. For instance, it is well-known that the Lie algebra of a Lie group endowed with bi-invariant pseudo-Riemannian metric turns out to be quadratic or –in terms of theoretical physics– that the Sugawara construction is possible for these Lie algebras [9], which shows its relevance in Conformal Field Theory. In [16] Medina and Revoy introduced the notion of double extension which results as the combination of a central extension and a semidirect product. This concept allowed them to give a certain inductive description of quadratic Lie algebras. More precisely, they proved that every quadratic Lie algebra may be constructed as a direct sum of irreducible ones, and the latter by a sequence of double extensions. A different construction, namely the T^*-extension, was given in [6] to describe all solvable quadratic Lie algebras (actually, the study in [6] is made not only in the case of Lie algebras, but for more general classes of algebras). The T^*-extension is essentially based on a generalized semidirect product. A different approach towards an inductive classification of quadratic Lie algebras has been recently given by Kath and Olbrich in [14].

The first attempt to use the technique of double extensions to describe quadratic Lie superalgebras was done in [3]. In this work the notion of double extension is defined for superalgebras, and it is shown that every irreducible quadratic Lie superalgebra in which the centre intersects the even part in a nontrivial way is a double extension. However, such a condition is rather restrictive and there do exist quadratic Lie superalgebras whose centre is contained in the odd part. One can even find nilpotent examples (see section 4 below) in which such a phenomenon occurs. Therefore, in order to give an inductive description of all quadratic Lie superalgebras there seems to be a need to generalize the concept of double extension. In this paper we shall introduce the notion of *generalized double extension* which is at the same time a generalization of the classical double extension and of the T^*-extension.

The paper is organized in 5 sections. The first one just recalls the basic definitions and preliminaries. In section 2 we introduce the concept of generalized double extension of quadratic Lie superalgebras and show that classical double extensions and T^*-extensions may be seen as particular cases. We then obtain a similar result to the one obtained by Medina and Revoy in the Lie algebra case; this is done in the third section in which we explicitly prove that every quadratic Lie algebra may be constructed by sucessive orthogonal sums and/or generalized double extensions. The particular case of solvable Lie superalgebras is studied in sections 4 and 5. We first see that every solvable quadratic Lie superalgebra may be obtained by a sequence of generalized double extensions by one-dimensional superalgebra and then, in Section 5, we give a description of such solvable quadratic superalgebras via the T^*-extension. We show that, in contrast to the case of the classical double extension, the T^*-extension allows us to describe all solvable cases.

We are indebted to M. Duflo for providing the proof of the central Theorem 5.1, which allowed us to extend a preliminary result concerning T^*-extensions which we had obtained (see [2]) to the general solvable case, and also for the second example of section 4. We also want to acknowledge the University of Vigo, the University of Metz, and the Graduiertenkolleg 'Partielle Differentialgleichungen' of the University of Freiburg for the reserch stays during which the main part of this work was done.

Notations: For a $\mathbb{Z}_2$-graded vector space V over the field $\mathbb{K}$ we write $V_{\overline{0}}$ for its even part and $V_{\overline{1}}$ for its odd part, i.e. $V = V_{\overline{0}} \oplus V_{\overline{1}}$. An element X of V is called homogeneous iff $X \in V_{\overline{0}}$ or $X \in V_{\overline{1}}$. In this work, all elements are supposed to be homogeneous unless otherwise stated. For a homogeneous element X we shall use the standard notation $|X| \in \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ to indicate its degree, i.e. whether it is contained in the even part ($|X| = \overline{0}$) or in the odd part ($|X| = \overline{1}$). Moreover, for two integers k, l we denote their images in $\mathbb{Z}_2$ by $\overline{k}, \overline{l}$, and we use the well-defined notation $(-1)^{\overline{k}\overline{l}}$ for $(-1)^{kl} \in \{-1, 1\}$.
All Lie superalgebras in this paper are finite-dimensional.

1 Preliminaries

Definition 1.1 Let $\mathfrak{g}$ be a Lie superalgebra and let B be a bilinear form on $\mathfrak{g}$.

i) B is called *supersymmetric* if $B(X,Y) = (-1)^{|X||Y|}B(Y,X), \quad \forall X \in \mathfrak{g}_{|X|}, \ \forall\, Y \in \mathfrak{g}_{|Y|}$.

ii) B is called *invariant* if $B([X,Y],Z) = B(X,[Y,Z]), \quad \forall\, X,Y,Z \in \mathfrak{g}$.

iii) B is called *even* if $B(X,Y) = 0, \quad \forall\, X \in \mathfrak{g}_{\overline{0}}, \ \forall\, Y \in \mathfrak{g}_{\overline{1}}$.

Definition 1.2 i) A Lie superalgebra $\mathfrak{g}$ is called *quadratic* if there exists a bilinear form B on $\mathfrak{g}$ such that B is supersymmetric, even, non-degenerate and invariant. In this case, B is called an *invariant scalar product* on $\mathfrak{g}$.

ii) Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra.

1) A graded ideal $\mathfrak{I}$ of $\mathfrak{g}$ is called non-degenerate (resp. degenerate) if the restriction of B to $\mathfrak{I} \times \mathfrak{I}$ is a non-degenerate (resp. degenerate) bilinear form.
2) The quadratic Lie superalgebra $(\mathfrak{g}, B)$ is called *irreducible* if $\mathfrak{g}$ contains no non-degenerate graded ideal other than $\{0\}$ and $\mathfrak{I}$.
3) We say that a graded ideal $\mathfrak{I}$ of $\mathfrak{g}$ is *irreducible* if $\mathfrak{I}$ is non-degenerate and $\mathfrak{I}$ contains no non-degenerate graded ideal of $\mathfrak{g}$ other than $\{0\}$ and $\mathfrak{I}$.
4) A graded ideal $\mathfrak{I}$ of $\mathfrak{g}$ will be said to be *isotropic* if $B(\mathfrak{I}, \mathfrak{I}) = \{0\}$.

Lemma 1.1 *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. Let $\mathfrak{I}$ be a graded ideal of $\mathfrak{g}$, we denote by $\mathfrak{I}^\perp$ the orthogonal space of $\mathfrak{I}$ with respect to B.*

i) $\mathfrak{I}^\perp$ is a graded ideal of $\mathfrak{g}$ and $[\mathfrak{I}, \mathfrak{I}^\perp] = \{0\}$.

ii) $\mathfrak{I}$ is non-degenerate if and only if $\mathfrak{I}^\perp$ is non-degenerate.

iii) If $[\mathfrak{g}, \mathfrak{I}] = \mathfrak{I}$, then $\mathfrak{I}^\perp = \mathcal{C}_{\mathfrak{g}}(\mathfrak{I})$, where $\mathcal{C}_{\mathfrak{g}}(\mathfrak{I})$ is the centralizer of $\mathfrak{I}$ in $\mathfrak{g}$. Moreover, $[\mathfrak{g}, \mathfrak{g}]^\perp = \mathfrak{z}(\mathfrak{g})$.

iv) If $\mathfrak{I}$ is non-degenerate, then $(\mathfrak{I}, \tilde{B} = B_{|\mathfrak{I}\times\mathfrak{I}})$ is a quadratic Lie superalgebra. Moreover the quadratic Lie superalgebra $(\mathfrak{I}, \tilde{B})$ is irreducible if and only if $\mathfrak{I}$ is an irreducible graded ideal of $\mathfrak{g}$.

v) If $\mathfrak{H}$ is a semisimple graded ideal of $\mathfrak{g}$, then $\mathfrak{H}$ is non-degenerate and $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{H}$.

The following Proposition reduces the study of quadratic Lie superalgebras to those having no non-degenerate graded ideal other than $\{0\}$ and $\mathfrak{I}$.

Proposition 1.2 *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. Then, $\mathfrak{g} = \oplus_{i=1}^{n}\mathfrak{g}_i$, where*

i) $\mathfrak{g}_i$ is a non-degenerate graded ideal, for all $i \in \{1, \dots, n\}$;

ii) $(\mathfrak{g}_i, B_i = B_{|\mathfrak{g}_i\times\mathfrak{g}_i})$ is a quadratic Lie superalgebra, for all $i \in \{1, \dots, n\}$;

iii) $B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$, for all $i, j \in \{1, \dots, n\}$.

A proof of both, Lemma 1.1 and Proposition 1.2, can be found for instance in [5].

2 Notion of generalized double extension of quadratic Lie superalgebras

¿From now on, if $\mathfrak{g}$ is a Lie superalgebra, we will denote by $\mathrm{Der}(\mathfrak{g})$ the Lie superalgebra of all its superderivations and by $\mathrm{End}(\mathfrak{g})$ that of its linear endomorphisms. The symbol $\sum_{cyclic}$ will be frequently used to denote cyclic sumation on a triple X, Y, Z.

2.1 Generalized semi-direct product of Lie superalgebras

Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ and $(\mathfrak{H}, [\ ,\ ]_{\mathfrak{H}})$ be two Lie superalgebras, $F : \mathfrak{g} \to \mathrm{Der}(\mathfrak{H})$ be an even linear map (not necessarily a morphism of Lie superalgebras), and $L : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{H}$ be an even superantisymmetric bilinear map such that the following equations are satisfied:

$$[F(X), F(Y)] - F([X, Y]_{\mathfrak{g}}) = \mathrm{ad}_{\mathfrak{H}} L(X, Y), \tag{1}$$

$$\sum_{cyclic} (-1)^{|X||Z|}\Big(F(X)\Big(L(X, Y)\Big) - L([X, Y]_{\mathfrak{g}}, Z)\Big) = 0,\ \forall (X, Y, Z) \in \mathfrak{g}_{|X|} \times \mathfrak{g}_{|Y|} \times \mathfrak{g}_{|Z|}. \tag{2}$$

We define the following product $[\ ,\ ]$ on the $\mathbb{Z}_2-$graded $\mathbb{K}-$vector space $\mathfrak{G} = \mathfrak{g} \oplus \mathfrak{H}$:

$$[X + h, Y + l] = [X, Y]_{\mathfrak{g}} + F(X)\Big(l\Big) - (-1)^{xy}F(Y)\Big(h\Big) + L(X, Y) + [h, l]_{\mathfrak{H}}, \tag{3}$$

for all $(X + h, Y + l) \in \mathfrak{G}_{|X|} \times \mathfrak{G}_{|Y|}$.

Using (1) and (2), we can see that the product $[\ ,\ ]$ defined a Lie superalgebra structure on $\mathfrak{G}$. The Lie superalgebra $\mathfrak{G}$ will be called the generalized semi-direct product of $\mathfrak{H}$ by $\mathfrak{g}$ by means of (F, L) (See [1] for more informations on this type of extension).

Remark 1 In particular, if $L = 0$ then $\mathfrak{G}$ is the semi-direct product of $\mathfrak{H}$ by $\mathfrak{g}$ by means of F (cf. [19], Chapter III, 1).

2.2 Generalized double extension of quadratic Lie superalgebras

Let $(\mathfrak{g}_1, [\ ,\ ]_1, B_1)$ a quadratic Lie superalgebra. We denote by $\mathrm{Der}_a(\mathfrak{g}_1)$ the Lie superalgebra of all superderivations of $\mathfrak{g}_1$ which are superantisymmetric with respect to B_1. Let $\mathrm{Out}_a(\mathfrak{g}_1)$ be the quotient of $\mathrm{Der}_a(\mathfrak{g}_1)$ by all inner derivations of $\mathfrak{g}_1$ which is known to be a Lie superalgebra. Let $(\mathfrak{g}_2, [\ ,\ ]_2)$ be an arbitrary Lie superalgebra of finite dimension and let B_2 be a supersymmetric invariant (not necessarily nondegenerate) bilinear form on $\mathfrak{g}_2$.
Let

$$\phi : \mathfrak{g}_2 \rightarrow \mathrm{Der}_a(\mathfrak{g}_1) \tag{4}$$

be an even linear map, and

$$\psi : \mathfrak{g}_2 \times \mathfrak{g}_2 \rightarrow \mathfrak{g}_1 \tag{5}$$

be an even bilinear superantisymmetric map which satisfy the following twisted morphism condition for all homogeneous elements $X, Y, Z \in \mathfrak{g}_2$:

$$[\phi(X), \phi(Y)] - \phi([X, Y]_2) = \mathrm{ad}_{\mathfrak{g}_1}(\psi(X, Y)). \tag{6}$$

Note that this condition implies that the composed map $\tilde{\phi} : \mathfrak{g}_2 \xrightarrow{\phi} \mathrm{Der}_a(\mathfrak{g}_1) \rightarrow \mathrm{Out}_a(\mathfrak{g}_1)$ is a morphism of Lie superalgebras.

Lemma 2.1 *With the above notation, the maps ϕ and ψ satisfy the following equation:*

$$\sum_{cyclic} (-1)^{|Z||X|} \mathrm{ad}_{\mathfrak{g}_2}\Big(\phi(X)(\psi(Y, Z)) - \psi([X, Y]_2, Z)\Big) = 0. \tag{7}$$

Proof. The Lemma easily follows by observing that the graded cyclic sum of $[[\phi(X), \phi(Y)], \phi(Z)]$ vanishes by means of the super Jacobi identity and by using twice equation (6).

This Lemma motivates the following, slightly stronger condition on ϕ and ψ:

$$\sum_{\text{cyclic}} (-1)^{|Z||X|} (\phi(X)(\psi(Y, Z)) - \psi([X, Y]_2, Z)) = 0. \tag{8}$$

Moreover, let

$$\chi : \mathfrak{g}_2 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_2^* \tag{9}$$

be the even bilinear map defined by the following equation for all homogeneous elements $X \in \mathfrak{g}_1$ and $A, B \in \mathfrak{g}_2$:

$$\chi(A, X)(B) := -(-1)^{|X||B|} B_1(\psi(A, B), X). \tag{10}$$

Denoting the *coadjoint representation of* $\mathfrak{g}_2$ on its dual by $A.F := -(-1)^{|A||F|} F \circ \mathrm{ad}_{\mathfrak{g}_2}(A)$ for all homogeneous elements $A \in \mathfrak{g}_2$ and $F \in \mathfrak{g}_2^*$, we finally need an even bilinear superantisymmetric map

$$w : \mathfrak{g}_2 \times \mathfrak{g}_2 \rightarrow \mathfrak{g}_2^* \tag{11}$$

satisfying a *twisted cocycle condition*

$$\sum_{\text{cyclic}} (-1)^{|C||A|} (A.w(B, C) + w(A, [B, C]_2) + \chi(A, \psi(B, C))) = 0. \tag{12}$$

and a *supercyclicity condition*

$$w(A, B)(C) = (-1)^{(|B|+|C|)|A|} w(B, C)(A), \tag{13}$$

for all homogeneous elements $A, B, C \in \mathfrak{g}_2$. Let finally

$$\Phi : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2^* \tag{14}$$

be the even superantisymmetric bilinear map defined by the following equation for all homogeneous elements $X, Y \in \mathfrak{g}_1$ and $A \in \mathfrak{g}_2$:

$$\Phi(X,Y)(A) := (-1)^{|A|(|X|+|Y|)} B_1(\phi(A)(X), Y). \tag{15}$$

We shall subsume these notions in a

Definition 2.1 Let $(\mathfrak{g}_1, B_1)$ be a quadratic Lie superalgebra, $\mathfrak{g}_2$ a Lie superalgebra and maps ϕ, ψ, w defined above satisfying the equations (6), (8), (12) and (13). We call $(\mathfrak{g}_1, B_1, \mathfrak{g}_2, \phi, \psi, w)$ a *context of generalized double extension* (of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the Lie superalgebra $\mathfrak{g}_2$.

We have the following

Lemma 2.2 *Let $(\mathfrak{g}_1, B_1, \mathfrak{g}_2, \phi, \psi, w)$ be a context of generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the Lie superalgebra $\mathfrak{g}_2$.*
Then the maps Φ and χ satisfy the following equations for all homogeneous elements $X, Y \in \mathfrak{g}_1$ and $A, B \in \mathfrak{g}_2$:

$$\Phi(\phi(A)X, Y) + (-1)^{|A||X|}\Phi(X, \phi(A)Y) - A.\Phi(X,Y) - \chi(A, [X,Y]_1) = 0, \tag{16}$$

and

$$\begin{aligned} \chi([A,B]_2, X) - \chi(A, \phi(B)X) + (-1)^{|A||B|}\chi(B, \phi(A)X) & \\ - A.\chi(B,X) + (-1)^{|A||B|}B.\chi(A,X) + \Phi(\psi(A,B), X) &= 0 \end{aligned} \tag{17}$$

Moreover, Φ is an even two-cocycle of the Lie superalgebra $\mathfrak{g}_1$ where ${\mathfrak{g}_2}^*$ is a trivial module for $\mathfrak{g}_1$, i.e. for three homogeneous elements $X, Y, Z \in \mathfrak{g}_1$ one has

$$\sum_{\text{cyclic}} (-1)^{|X||Z|}\Phi(X, [Y,Z]) = 0. \tag{18}$$

Proof. Using the definitions of the maps ϕ, ψ, χ, and Φ the first two equations follow in a straight forward manner from the defining conditions (6) and (8), respectively. The cocycle condition is a consequence of the fact that $\phi(\mathfrak{g}_2)$ is a subset of $\mathrm{Der}_a(\mathfrak{g}_1)$ and has been shown in [3].

We are now in the position to put the structure of a quadratic Lie superalgebra on the vector space $\mathfrak{g} := \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^*$:

Theorem 2.3 *Let $(\mathfrak{g}_1, B_1, \mathfrak{g}_2, \phi, \psi, w)$ be a context of generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the Lie superalgebra $\mathfrak{g}_2$.*

Let $A, B \in \mathfrak{g}_2$, $X, Y \in \mathfrak{g}_1$, and $F, H \in \mathfrak{g}_2^*$ homogeneous elements such that $|A| = |X| = |F|$ and $|B| = |Y| = |H|$. We define the following bracket $[\ ,\ ]$ on the vector space $\mathfrak{g}$:

$$\begin{aligned} [A+X+F, B+Y+H] \;:=\; & [A,B]_2 \\ & +[X,Y]_1 + \Phi(X,Y) + \psi(A,B) + \phi(A)(Y) - (-1)^{|A||B|}\phi(B)(X) \\ & +A.H - (-1)^{|A||B|}B.F + \chi(A,Y) - (-1)^{|A||B|}\chi(B,X) \\ & \quad + w(A,B), \end{aligned} \tag{19}$$

and the following bilinear form B on $\mathfrak{g}$:

$$B(A+X+F,B+Y+H) := B_1(X,Y)+F(B)+(-1)^{|A||B|}H(A). \tag{20}$$

Then the triple $(\mathfrak{g},[\ ,\ ],B)$ is a quadratic Lie superalgebra such that the subspace $\mathfrak{g}_2^*$ is an isotropic ideal of $\mathfrak{g}$ and the subspace $\mathfrak{g}_1 \oplus \mathfrak{g}_2^*$ is the orthogonal space of $\mathfrak{g}_2^*$.

The quadratic Lie superalgebra $\mathfrak{g}$ is called the generalized double extension *of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the Lie superalgebra $\mathfrak{g}_2$ by means of (ϕ,ψ,w).*

Proof. By Lemma 2.2 we have $\Phi \in (Z^2(\mathfrak{g}_1,\mathfrak{g}_2^*))_{\bar{0}}$, it follows that we can define the following Lie superalgebra structure on the $\mathbb{K}-$vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2^*$:

$$[X+F,Y+H] = [X,Y]_{\mathfrak{g}_1}+\Phi(X,Y),\ \forall\ X+F \in (\mathfrak{g}_1\oplus\mathfrak{g}_2^*)_{|X|}, Y+H \in (\mathfrak{g}_1\oplus\mathfrak{g}_2^*)_{|Y|}.$$

This Lie superalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2^*$ is the central extension of $\mathfrak{g}_1$ by $\mathfrak{g}_2^*$ by means of Φ.
Now, if A is an homogeneous element of $\mathfrak{g}_2$ we consider $\tilde{\phi}(A) \in (\mathrm{End}(\mathfrak{g}_1\oplus\mathfrak{g}_2^*))_{|A|}$ defined by:

$$\tilde{\phi}(A)(X+F) := \phi(A)(X)+\pi(A)(F)+\chi(A,X),\ \forall A+F \in \mathfrak{g}_1\oplus\mathfrak{g}_2^*, \tag{21}$$

where π is the coadjoint representation of $\mathfrak{g}_2$. It is easy to see that equation (16) and the fact that $\phi(A) \in \mathrm{Der}(\mathfrak{g}_1)$ imply that $\tilde{\phi}(A)$ is a superderivation of Lie superalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2^*$. It follows that we have the even linear map

$$\tilde{\phi} : \mathfrak{g}_2 \to \mathrm{Der}(\mathfrak{g}_1\oplus\mathfrak{g}_2^*)$$

defined by (21) on the homogeneous elements of $\mathfrak{g}_2$.
Let us consider the even bilinear superantisymmetric map

$$\tilde{\psi} : \mathfrak{g}_2\times\mathfrak{g}_2 \to \mathfrak{g}_1\oplus\mathfrak{g}_2^*$$

defined by:

$$\tilde{\psi}(A,B) := \psi(A,B)+w(A,B),\ \forall A \in (\mathfrak{g}_2)_{|A|}, B \in (\mathfrak{g}_2)_{|B|}.$$

By the equations (6), (17) we have

$$[\tilde{\phi}(A),\tilde{\phi}(B)]-\tilde{\phi}([A,B]_2) = \mathrm{ad}_{\mathfrak{g}_1\oplus\mathfrak{g}_2^*}(\tilde{\psi}(A,B)),\ \forall A \in (\mathfrak{g}_2)_{|A|}, B \in (\mathfrak{g}_2)_{|B|}. \tag{22}$$

Moreover, equations (8), (12) imply that

$$\sum_{cyclic}(-1)^{|A||C|}\Big(\tilde{\phi}(A)\Big(\tilde{\psi}(B,C)\Big)-\tilde{\psi}([A,B]_{\mathfrak{g}},C)\Big) = 0,\ \forall (A,B,C) \in (\mathfrak{g}_2)_{|A|}\times(\mathfrak{g}_2)_{|B|}\times(\mathfrak{g}_2)_{|C|}. \tag{23}$$

Consequently, equations (22), (23) imply that we can consider the generalized semi-direct product $\mathfrak{g} := \mathfrak{g}_2\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2^*$ of $\mathfrak{g}_1\oplus\mathfrak{g}_2^*$ by $\mathfrak{g}_2$ by means of $(\tilde{\phi},\tilde{\psi})$. It is easy to verify that the bracket of $\mathfrak{g}$ is exactly given by the bracket $[\ ,\ ]$ of equation (19) defined in Theorem 2.3, and the bilinear form B as defined in (20) is an invariant scalar product on $\mathfrak{g}$. Finally, it is clear that the subspace $\mathfrak{g}_2^*$ is an isotropic ideal of $\mathfrak{g}$, and that the subspace $\mathfrak{g}_1\oplus\mathfrak{g}_2^*$ is the orthogonal space of $\mathfrak{g}_2^*$.

Remark 2 If $\mathfrak{g}$ is the generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the Lie superalgebra $\mathfrak{g}_2$ by means of (ϕ,ψ,w) and if B_2 is a supersymmetric invariant (not necessarily nondegenerate) bilinear form on $\mathfrak{g}_2$, then the following bilinear form B' on $\mathfrak{g}$:

$$B'(A+X+F,C+Y+H) := B_2(A,C)+B_1(X,Y)+F(C)+(-1)^{|A||C|}H(A), \tag{24}$$

is another invariant scalar product on $\mathfrak{g}$.

2.3 Double extension and T^*-extension: Important particular cases

Two particular cases of the generalized double extension should be mentioned: double extension and T^*-extensions. We first recall the notion of double extension of quadratic Lie superalgebras as given in [3]:

Theorem 2.4 *Let $(\mathfrak{g}_1, B_1)$ be a quadratic Lie superalgebra, $\mathfrak{g}_2$ a Lie superalgebra and $\phi : \mathfrak{g}_2 \to \mathrm{Der}_a(\mathfrak{g}_1) \subset \mathrm{Der}(\mathfrak{g}_1)$ a morphism of Lie superalgebras.*
Let ψ be the map from $\mathfrak{g}_1 \times \mathfrak{g}_1$ to $\mathfrak{g}_2^*$, defined by:

$$\psi(X, Y)(Z) = (-1)^{(|X|+|Y|)|Z|} B_1(\psi(Z)(X), Y) \qquad \forall X \in (\mathfrak{g}_1)_{|X|},\ \forall Y \in (\mathfrak{g}_1)_{|Y|},\ \forall Z \in (\mathfrak{g}_2)_{|Z|}.$$

Let π be the coadjoint representation of $\mathfrak{g}_2$. Then the $\mathbb{Z}_2-$graded vector space $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus {\mathfrak{g}_2}^*$ with the product defined by

$$\begin{aligned} [X_2 + X_1 + f, Y_2 + Y_1 + g] &= [X_2, Y_2]_{\mathfrak{g}_2} + [X_1, Y_1]_{\mathfrak{g}_1} + \phi(X_2)(Y_1) - (-1)^{|X||Y|}\phi(Y_2)(X_1) \\ &\quad + \pi(X_2)(g) - (-1)^{|X||Y|}\pi(Y_2)(f) + \psi(X_1, Y_1), \end{aligned}$$

(where $|X| = |X_2| = |X_1| = |f|$ and $|Y| = |Y_2| = |Y_1| = |g|$) is a Lie superalgebra.

Moreover, if γ is an invariant supersymmetric bilinear form on $\mathfrak{g}_2$, then the bilinear form T defined on $\mathfrak{g}$ by

$$T(X_2 + X_1 + f, Y_2 + Y_1 + g) = B_1(X_1, Y_1) + \gamma(X_2, Y_2) + f(Y_2) + (-1)^{|X||Y|} g(X_2)$$

is an invariant scalar product on $\mathfrak{g}$.

The Lie superalgebra $\mathfrak{g}$ is called a double extension of $(\mathfrak{g}_1, B_1)$ by $\mathfrak{g}_2$ by means of ϕ.

Let $(\mathfrak{g}_1, B_1, \mathfrak{g}_2, \phi, \psi, w)$ be a context of generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the Lie superalgebra $\mathfrak{g}_2$. If $\psi = 0, w = 0$ then the generalized double extension $\mathfrak{g}$ of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the Lie superalgebra $\mathfrak{g}_2$ by means of (ϕ, ψ, w) is actually the double extension of $(\mathfrak{g}_1, B_1)$ by $\mathfrak{g}_2$ by means ϕ in the sense of the Theorem.

Now, let $(\mathfrak{g}_1, B_1, \mathfrak{g}_2, \phi, \psi = 0, w)$ be a context of generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1 = 0, B_1 = 0)$ by the Lie superalgebra $\mathfrak{g}_2$. Let $\mathfrak{g}$ be the generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1 = 0, B_1 = 0)$ by the Lie superalgebra $\mathfrak{g}_2$ by means of $(\phi = 0, \psi = 0, w)$. The superalgebra $\mathfrak{g}$ will be called the *T^*-extension of $\mathfrak{g}$ by means of w*, and we shall denote $\mathfrak{g}$ by the symbol $T^*_w\mathfrak{g}$ or $T^*\mathfrak{g}$. In this case, by Theorem 2.3, we have the following Proposition.

Proposition 2.5 *Let $\mathfrak{g}_2$ be a Lie superalgebra, $\mathfrak{g}_2^*$ its dual space, and π its coadjoint representation. Let w be an even 2-cocycle of $\mathfrak{g}_2$ with values in ${\mathfrak{g}_2}^*$. Define the following structures on the vector space $\mathfrak{g} := \mathfrak{g}_2 \oplus \mathfrak{g}_2^*$: For a pair of homegeneous elements $X + F, Y + H$ of $\mathfrak{g}$ of degree $|X| = |F|$ (resp.$|Y| = |H|$), let*

$$[X + F, Y + H]_{\mathfrak{g}} := [X, Y]_{\mathfrak{g}} + w(X, Y) + \pi(X)(H) - (-1)^{|X||Y|}\pi(Y)(F),$$

and

$$B(X + F, Y + H) := F(Y) \ + \ (-1)^{|X||Y|} \ H(X).$$

Then $\mathfrak{g}$ endowed with the product $[\,,\,]_{\mathfrak{g}}$ is a Lie superalgebra.

Moreover, $(\mathfrak{g}, B)$ is a quadratic Lie superalgebra if and only if w is supercyclic, i.e.

$$w(X,Y)(Z) = \ (-1)^{|X|(|Y|+|Z|)}\ w(Y,Z)(X), \quad \forall\ (X,Y,Z) \in \mathfrak{g}_{|X|} \times \mathfrak{g}_{|Y|} \times \mathfrak{g}_{|Z|}.$$

We shall speak of the quadratic Lie superalgebra $(\mathfrak{g}, B)$ constructed out of $\mathfrak{g}$ and the supercyclic 2-cocycle w as a T^*-extension of $\mathfrak{g}$ by means w *and shall denote $\mathfrak{g}$ by the symbol $T^*_w\mathfrak{g}$ or $T^*\mathfrak{g}$.*

Remark 3 It is clear from the definition that if $\mathfrak{g} = T^*_w(\mathfrak{g}_2)$, then its dimension $n = \dim(\mathfrak{g})$ is even, and $\mathfrak{I} = \mathfrak{g}_2^*$ is an isotropic graded ideal of dimension $n/2$. Further, as a consequence of Theorem 3.1 below, it is not difficult to see that the fact of being even-dimensional and the existence of a graded ideal $\mathfrak{I}$ such that $\mathfrak{I} = \mathfrak{I}^{\perp}$ characterize T^*-extensions.

It should be noticed that if w is a supercyclic even 2-cocycle with values in $\mathfrak{g}_2^*$ the the trilinear mapping $\hat{w} : (\mathfrak{g}_2)^3 \to \mathbb{K}$ given by $\hat{w}(X,Y,Z) = w(X,Y)Z$, turns out to be an even scalar 3-cocycle. Actually, one gets in this way an isomorphism between the subspace of $(Z^2(\mathfrak{g}_2, \mathfrak{g}_2^*))_{\bar{0}}$ composed of all supercyclic elements and $(Z^3(\mathfrak{g}_2, \mathbb{K}))_{\bar{0}}$. Therefore one easily gets the following result which gives a certain classification of the T^*-extensions of a given Lie algebra in terms of the cohomology class of $\hat{w}$ (see [2]).

Proposition 2.6 *Let $\mathfrak{g}_2$ be a Lie superalgebra, w_1, w_2 two supercyclic even 2-cocycles with values in $\mathfrak{g}_2^*$, and consider an even superantisymmetric bilinear mapping $\varphi : \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathbb{K}$.*
The map $S_\varphi : T^*_{w_1}(\mathfrak{g}_2) \to T^*_{w_2}(\mathfrak{g}_2)$ defined by $S_\varphi(X + F) = X + \varphi(X, \cdot) + F$ defines an isometry of quadratic Lie superalgebras if and only if $\hat{w}_1 - \hat{w}_2$ is a coboundary.

3 Inductive description of quadratic Lie superalgebras

In this section, we are going to give an inductive description of quadratic Lie superalgebras by using the notion of generalized double extension of quadratic Lie superalgebras.

Theorem 3.1 *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra, $\mathfrak{I}$ be an isotropic graded ideal of $\mathfrak{g}$, and $\mathfrak{I}^{\perp}$ its orthogonal space. Define*

1. *the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ with $\mathfrak{g}_1 := \mathfrak{I}^{\perp}/\mathfrak{I}$ and B_1 the quadratic form induced by the restriction of B to $\mathfrak{I}^{\perp} \times \mathfrak{I}^{\perp}$, and*

2. *the Lie superalgebra $\mathfrak{g}_2 := \mathfrak{g}/\mathfrak{I}^{\perp}$.*

Then for any isotropic graded vector subspace $\mathcal{V}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{I}^{\perp} \oplus \mathcal{V}$ and any graded subspace $\mathcal{A}$ of $\mathfrak{I}^{\perp}$ with $\mathfrak{I}^{\perp} = \mathcal{A} \oplus \mathfrak{I}$ there is an even linear map $\phi : \mathfrak{g}_2 \to \mathrm{Der}_a(\mathfrak{g}_1)$, an even bilinear superantisymmetric map $\psi : \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathfrak{g}_1$, and an even bilinear superantisymmetric map $w : \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathfrak{g}_2^*$ such that $(\mathfrak{g}, B)$ is isometric to the generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by $\mathfrak{g}_2$ by means of (ϕ, ψ, w).

Proof. It is clear that $\mathfrak{I} \subseteq \mathfrak{I}^{\perp}$ since $\mathfrak{I}$ is isotropic. Choose a graded vector subspace $\mathcal{A}$ of $\mathfrak{I}^{\perp}$ such that $\mathfrak{I}^{\perp} = \mathcal{A} \oplus \mathfrak{I}$. Since $\mathfrak{I}$ is the orthogonal space of $\mathfrak{I}^{\perp}$ it is obvious that $\mathcal{A}$ is nondegenerate. Hence $\mathcal{A}^{\perp}$ is nondegenerate, and we have $\mathcal{A}^{\perp} \cap \mathfrak{I}^{\perp} = \mathfrak{I}$. Moreover, since $\mathcal{A} \cap \mathfrak{I} = \{0\}$ it follows

that $\mathfrak{g} = \mathcal{A}^\perp + \mathfrak{I}^\perp$ Choose a graded vector subspace $\mathcal{V}$ of $\mathcal{A}^\perp \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{I}^\perp \oplus \mathcal{V}$. Hence $\mathcal{A}^\perp = \mathcal{V} \oplus \mathfrak{I}$. Since the field $\mathbb{K}$ is algebraically closed and of characteristic different from 2 we may choose the graded subspace $\mathcal{V}$ to be isotropic.

For any $A \in \mathfrak{g} = \mathcal{V} \oplus \mathcal{A} \oplus \mathfrak{I}$ denote by $A_\mathcal{V}$ its component in $\mathcal{V}$, by $A_\mathcal{A}$ its component in $\mathcal{A}$, by $A_\mathfrak{I}$ its component in $\mathfrak{I}$, and by $A_{\mathfrak{I}^\perp}$ its component in $\mathfrak{I}^\perp$. Clearly, $A = A_\mathcal{V} + A_\mathcal{A} + A_\mathfrak{I} = A_\mathcal{V} + A_{\mathfrak{I}^\perp}$. Moreover, let $\mathfrak{I}^\perp \to \mathfrak{g}_1 : X \mapsto \overline{X}$ be the natural projection which is a morphism of Lie superalgebras, and let $K : \mathcal{V} \to \mathfrak{g}_2$ be the even linear map $\mathcal{V} \hookrightarrow \mathfrak{g} \to \mathfrak{g}_2$ which clearly is a linear isomorphism satisfying

$$K([V_1, V_2]_\mathcal{V}) = [KV_1, KV_2]_{\mathfrak{g}_1}$$

for all $V_1, V_2 \in \mathcal{V}$. Using this notation we shall define the following three even maps

$$\begin{array}{rcl} \phi : \mathfrak{g}_2 \to \mathrm{Hom}_\mathbb{K}(\mathfrak{g}_1, \mathfrak{g}_1) & : & KV \mapsto (\overline{X} \mapsto \overline{[V, X]}), \\ \psi : \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathfrak{g}_1 & : & (KV_1, KV_2) \mapsto \overline{[V_1, V_2]_{\mathfrak{I}^\perp}}, \\ w : \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathfrak{g}_2^* & : & (KV_1, KV_2) \mapsto (KV_3 \mapsto B([V_1, V_2], V_3)), \end{array}$$

which are well-defined since K is a linear isomorphism and $\mathfrak{I}^\perp$ is a graded ideal of $\mathfrak{g}$.
Since the ideals $\mathfrak{I}^\perp$ and $\mathfrak{I}$ obviously are $\mathfrak{g}$-modules by means of the adjoint representation, it follows that their quotient $\mathfrak{g}_1$ is a $\mathfrak{g}$-module on which $\mathfrak{g}$ acts by graded derivations. Therefore the values of the linear map ϕ are graded derivations of $\mathfrak{g}_1$, a fact which also follows directly by the graded Jacobi identity of $\mathfrak{g}$. Moreover since the induced quadratic form B_1 on $\mathfrak{g}_1$ is of the form $B_1(\overline{X}, \overline{Y}) := B(X, Y)$ for all $X, Y \in \mathfrak{I}^\perp$ we have for all $V \in \mathcal{V}$

$$\begin{array}{rcl} B_1(\phi(KV)(\overline{X}), \overline{Y}) & = & B_1(\overline{[V, X]}, \overline{Y}) = B([V, X], Y) = -(-1)^{|V||X|} B(X, [V, Y]) \\ & = & -(-1)^{|V||X|} B_1(\overline{X}, \phi(KV)(\overline{Y})), \end{array}$$

whence ϕ takes its values in $\mathrm{Der}_a(\mathfrak{g}_1)$ and satisfies the defining condition (4).
Moreover, ψ is clearly an even superantisymmetric bilinear map. The fact that $[V_1, V_2] = [V_1, V_2]_\mathcal{V} + [V_1, V_2]_{\mathfrak{I}^\perp}$ and the graded Jacobi identity on three elements $V_1, V_2, V_3 \in \mathcal{V}$ show that condition (8) is satisfied on ϕ and ψ.
Finally, it is obvious that w is an even superantisymmetric bilinear map. The supercyclic condition (13) for w follows from the invariance of B. Furthermore, for $V_1, V_2, V_3, V \in \mathcal{V}$ we get for the map χ (see equation (9)):

$$\begin{array}{rcl} \chi(KV_1, \psi(KV_2, KV_3))(KV) & = & -(-1)^{(|V_2|+|V_3|)|V|} B_1(\psi(KV_1, KV), \psi(KV_2, KV_3)) \\ & = & -(-1)^{(|V_2|+|V_3|)|V|} B([V_1, V]_{\mathfrak{I}^\perp}, [V_2, V_3]_{\mathfrak{I}^\perp}) \end{array}$$

whence for one of the cyclic terms in condition (12) we get

$$\begin{array}{l} \Big((KV_1).(w(KV_2, KV_3))\Big)(KV) + w(KV_1, [KV_2, KV_3])(KV) + \chi(KV_1, \psi(KV_2, KV_3))(KV) = \\ \quad -(-1)^{(|V_2|+|V_3|)|V_1|}\Big(B([V_2, V_3], [V_1, V]_\mathcal{V}) + B([V_2, V_3]_\mathcal{V}, [V_1, V]) + B([V_2, V_3]_{\mathfrak{I}^\perp}, [V_1, V]_{\mathfrak{I}^\perp})\Big) \\ \quad = -(-1)^{(|V_2|+|V_3|)|V_1|}\Big(B([V_2, V_3], [V_1, V])\Big) \end{array}$$

since $B([V_2, V_3]_\mathcal{V}, [V_1, V]) = B([V_2, V_3]_\mathcal{V}, [V_1, V]_{\mathfrak{I}^\perp})$ due to the fact that $\mathcal{V}$ is isotropic. The graded cyclic sum of the preceding terms vanishes, and thus condition (12) holds thanks to the invariance of B and the graded Jacobi identity in $\mathfrak{g}$.

It follows that $\left(\mathfrak{g}_1, B_1, \mathfrak{g}_2, \phi, \psi, w\right)$ is a context of generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1 := \mathfrak{I}^\perp/\mathfrak{I}, B_1)$ by the Lie superalgebra $\mathfrak{g}_2$ which is isomorphic to $\mathcal{V}$ as a graded vector space. Consider the generalized double extension $\tilde{\mathfrak{g}} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^*$ of $(\mathfrak{g}_1, B_1)$ by $\mathfrak{g}_2$ by means of (ϕ, ψ, w), we denote by $\tilde{B}$ its invariant scalar product defined in Theorem 2.3. Let $\nabla : \mathfrak{I} \to \mathfrak{g}_2^*$ be the even linear map

$$\nabla(X)(KV) := B(X, V).$$

It is easy to verify in a long, but straightforward manner that the following linear map

$$\Pi : \mathfrak{g} = \mathcal{V} \oplus \mathcal{A} \oplus \mathfrak{I} \to \tilde{\mathfrak{g}} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^*$$

defined by

$$\Pi(V + A + X) := KV + \overline{A} + \nabla(X)$$

for all $V \in \mathcal{V}$, $A \in \mathcal{A}$, and $X \in \mathfrak{I}$ is an isomorphism of Lie superalgebras. Moreover,

$$\tilde{B}(\Pi(X), \Pi(Y)) = B(X, Y), \quad \forall\, X, Y \in \mathfrak{g},$$

i.e. Π is an isometry, which proves the Theorem.

Corollary 3.2 *Let $(\mathfrak{g}, B)$ be an irreducible quadratic Lie superalgebra which is neither simple nor the one-dimensional Lie algebra. Then, $(\mathfrak{g}, B)$ is a generalized double extension of a quadratic Lie superalgebra $(\mathcal{A}, T)$ by a simple Lie superalgebra or by the one-dimensional Lie algebra or by the one-dimensional odd Lie superalgebra $\mathcal{N} = \mathcal{N}_{\bar{1}}$.*

Proof. Since $\mathfrak{g}$ is neither a simple nor the one-dimensional Lie algebra, then there exists a non-zero minimal graded ideal $\mathfrak{I}$ of $\mathfrak{g}$. The fact that $(\mathfrak{g}, B)$ is an irreducible quadratic Lie superalgebra implies that $\mathfrak{I}$ is isotropic. Therefore, by Theorem 3.1, $(\mathfrak{g}, B)$ is a generalized double extension of a quadratic Lie superalgebra $(\mathfrak{I}^\perp/\mathfrak{I}, B_1)$ by the Lie superalgebra $\mathfrak{g}/\mathfrak{I}^\perp$. Since $\mathfrak{I}$ is a minimal graded ideal of $\mathfrak{g}$, then $\mathfrak{I}^\perp$ is a maximal graded ideal of $\mathfrak{g}$. Consequently, $\mathfrak{g}/\mathfrak{I}^\perp$ is either a simple Lie superalgebra or the one-dimensional Lie algebra or the one-dimensional odd Lie superalgebra $\mathcal{N} = \mathcal{N}_{\bar{1}}$.

Let $\mathcal{B}$ be the set consisting of $\{0\}$, the isometry classes of quadratic simple Lie superalgebras, and the one-dimensional Lie algebra.

Theorem 3.3 *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. Then, either $\mathfrak{g}$ is an element of $\mathcal{B}$ or $(\mathfrak{g}, B)$ is obtained by a sequence of generalized double extensions by a simple Lie superalgebra or by the one-dimensional Lie algebra or by the one-dimensional Lie superalgebra $\mathcal{N} = \mathcal{N}_{\bar{1}}$ and/or orthogonal direct sums of quadratic Lie superalgebras from a finite number of elements of $\mathcal{B}$.*

Remark 4 It can be easily seen that the Cartan superalgebra $W(n) = \mathrm{Der}(\bigwedge V)$, where V is a n-dimensional vector space, is not quadratic for $n \geq 3$ since for every even bilinear supersymmetric and invariant form B on $W(n)$ the non-null subspace $W_{n-1} = \{D \in W(n) \,|\, D(V) \subset \bigwedge^n V\}$ is orthogonal with respect to B to the whole $W(n)$.

In Proprosition 3 of [19, page 187] it is proved that if V is a n-dimensional vector space with $n \geq 4$ then the gradation $S(V) = \oplus_{r=-1}^{n-2} S_r(V)$ verifies $[S_r(V), S_1(V)] = S_{r+1}(V)$ for all $r \geq -1$.

Therefore if B is an even invariant bilinear supersymmetric form then $B(S_{n-2}(V), S(V)) = \{0\}$, which proves that B is degenerate and hence $S(V)$ cannot be quadratic.

The same reasoning shows that $H(n)$ is not quadratic for $n \geq 5$ since from Proprosition 7 in [19, page 196] one has $[H_r(V), H_1(V)] = H_{r+1}(V)$ for all $r \geq -1$ and hence H_{n-3} must be orthogonal to $H(n)$ with respect to every even invariant bilinear supersymmetric form.

We conclude that, with the possible exception of the superalebras $\tilde{S}(2r)$, $r \geq 1$, a non-classical simple Lie superalgebra cannot be quadratic.

4 Solvable quadratic Lie superalgebras are generalized double extensions by one-dimensional Lie superalgebras

It has been proved in [3] that every n-dimensional irreducible quadratic Lie superalgebra $\mathfrak{g}$ such that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$ is a double extension of a $(n-2)$-dimensional quadratic Lie superalgebra by a one-dimensional ideal. However, the condition $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$ is not always satisfied (even in the nilpotent case) as the following examples show:

Examples

1. For each positive integer n, let $\mathcal{T}(n)$ denote the linear space of square matrices of order n which are upper triangular and $\mathcal{N}(n)$ its subspace of strict upper triangular matrices. It is easy to prove that

$$\mathfrak{g}(n) = \left\{ \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \middle| \, A, C, D \in \mathcal{N}(n), B \in \mathcal{T}(n) \right\}$$

is a Lie sub-superalgebra of $\mathfrak{gl}(n,n)$ and that $\dim(\mathfrak{z}(\mathfrak{g}(n))) = 1$, $\mathfrak{z}(\mathfrak{g}(n)) \subset \mathfrak{g}(n)_{\bar{1}}$, and $[\mathfrak{g}(n)_{\bar{1}}, \mathfrak{g}(n)_{\bar{1}}] = \mathfrak{g}(n)_{\bar{0}}$. Let $\mathcal{E}_n = T_0^*(\mathfrak{g}(n))$ be the T^*-extension of $\mathfrak{g}(n)$ by $w = 0$. It is clear that $\mathcal{E}_n$ is nilpotent since so is $\mathfrak{g}(n)$. Its centre is given by

$$\mathfrak{z}(\mathcal{E}_n) = \mathfrak{z}(\mathfrak{g}(n)) \oplus \{f \in \mathfrak{g}(n)^* \,|\, f([\mathfrak{g}(n), \mathfrak{g}(n)]) = \{0\}\}$$

and since $[\mathfrak{g}(n)_{\bar{1}}, \mathfrak{g}(n)_{\bar{1}}] = \mathfrak{g}(n)_{\bar{0}}$ one easily gets that $\{0\} \neq \mathfrak{z}(\mathcal{E}_n) \subset \mathfrak{g}(n)_{\bar{1}} \oplus \mathfrak{g}(n)^*_{\bar{1}} = (\mathcal{E}_n)_{\bar{1}}$.

2. The superalgebras constructed in the example above have dimension $2n(2n-1)$ and hence, the smallest one is the 12-dimensional superalgebra $\mathcal{E}_2$. M. Duflo drew our attention to the following 7-dimensional example: Consider the quadratic superspace V where the even part is one-dimensional and the odd part has dimension 2 and let $\mathcal{N}$ be the standard nilpotent sub-superalgebra of $\mathfrak{osp}(1,2)$. The superalgebra double extension of V by $\mathcal{N}$ has also its centre contained in the odd part.

Obviously, the results in [3] cannot be applied in these examples. Further, one can see that the second one gives a quadratic Lie superalgebra which cannot be constructed as a (classical) double extension by a one-dimensional algebra. Thus, the inductive classification of solvable quadratic Lie superalgebras requires the use of generalized double extensions. We will prove in this section that, actually, every such superalgebra may be obtained by sucessive generalized double extensions by one-dimensional superalgebras.

It is easy to verify that $(\mathfrak{g}_1, B_1, \mathfrak{g}_2 := \mathcal{N}, \phi, \psi, w)$ is a context of generalized double extension of a quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the one-dimensional Lie superalgebra $\mathcal{N} = \mathcal{N}_{\bar{1}} = \mathbb{K}e$, if and only if $w = 0$ and there exists $(D, X_0) \in [\mathrm{Der}_a(\mathfrak{g}_1, B_1)]_{\bar{1}} \times (\mathfrak{g}_1)_{\bar{0}}$ such that:

$$\phi(e) = D,\ \psi(e, e) = X_0,\ D(X_0) = 0,\ B_1(X_0, X_0) = 0 \text{ and } D^2 = \frac{1}{2}\mathrm{ad}_{\mathfrak{g}_1} X_0.$$

If $\mathfrak{g}$ is the generalized double extension of the quadratic Lie superalgebra $(\mathfrak{g}_1, B_1)$ by the Lie superalgebra $\mathfrak{g}_2 := \mathcal{N}$ by means of $(\phi, \psi, w := 0)$, then the bracket $[\,,\,]$ on $\mathfrak{g} = \mathbb{K}e \oplus \mathfrak{g}_1 \oplus \mathbb{K}e^*$ (where $\{e^*\}$ is the dual basis of $\{e\}$) is defined by:

$$[e, e] = X_0,\ [e^*, \mathfrak{g}] = \{0\},$$

$$[e, X] = D(X) - B_1(X, X_0)e^*,$$

$$[X, Y] = [X, Y]_{\mathfrak{g}_1} - B(D(X), Y)e^*,\ \forall X \in \mathfrak{g}_{|X|},\ Y \in \mathfrak{g},$$

and the invariant scalar product B on $\mathfrak{g}$ is defined by:

$$B(e^*, e) = 1,\ B_{\mathfrak{g}_1 \times \mathfrak{g}_1} := B_1,$$

$$B(x, e) = B(x, e^*) = B(e^*, e^*) = B(e, e) = 0,\ \forall X \in \mathfrak{g}_1.$$

Proposition 4.1 *Let $(\mathfrak{g}, B)$ be an irreducible quadratic Lie superalgebra. If $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} \neq \{0\}$, then $(\mathfrak{g}, B)$ is a generalized double extension of a quadratic Lie superalgebra $(\mathcal{A}, T)$ by the one-dimensional Lie superalgebra $\mathcal{N} = \mathcal{N}_{\bar{1}}$.*

Proof. Suppose that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} \neq \{0\}$. Let $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} \backslash \{0\}$. Then $\mathfrak{I} = \mathbb{K}X$ is an isotropic graded ideal of $\mathfrak{g}$. Therefore there exists $Y \in \mathfrak{g}_{\bar{1}}$ such that $B(X, Y) = 1$, and $B(Y, Y) = 0$. It follows that $\mathfrak{g} = \mathfrak{I}^{\perp} \oplus \mathbb{K}Y$. Then, by Theorem 3.1, $(\mathfrak{g}, B)$ is a generalized double extension of a quadratic Lie superalgebra $(\mathcal{A}, T)$ by the one-dimensional Lie superalgebra $\mathcal{N} = \mathcal{N}_{\bar{1}}$.

Corollary 4.2 *Let $(\mathfrak{g}, B)$ be an irreducible quadratic Lie superalgebra such that $\mathfrak{g}$ is not the one-dimensional Lie algebra. If $\mathfrak{g}$ is solvable, then either $(\mathfrak{g}, B)$ is a double extension of a solvable quadratic Lie superalgebra $(\mathcal{A}, T)$ by the one-dimensional Lie algebra or $(\mathfrak{g}, B)$ is a generalized double extension of a solvable quadratic Lie superalgebra $(\mathcal{H}, U)$ by the one-dimensional Lie superalgebra.*

Proof. Since $\mathfrak{g}$ is solvable, then $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$. If $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} \neq \{0\}$, (resp. $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$), then, by Corollary 4.1 (resp. by [3], Corollary 1 of Proposition 3.2.3), $(\mathfrak{g}, B)$ is a generalized double extension of a quadratic Lie superalgebra $(\mathcal{H}, U)$ by the one-dimensional Lie superalgebra (resp. $(\mathfrak{g}, B)$ is a double extension of a quadratic Lie superalgebra $(\mathcal{A}, T)$ by the one-dimensional Lie algebra) where $\mathcal{H} = \mathfrak{I}^{\perp}/\mathfrak{I}$ (resp. $\mathcal{A} = \mathfrak{I}^{\perp}/\mathfrak{I}$) with $\mathfrak{I} = \mathbb{K}X$ where $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} \backslash \{0\}$ (resp. $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \backslash \{0\}$). The fact that $\mathfrak{g}$ is solvable implies that its graded ideal is also solvable, it follows that the Lie superalgebra $\mathcal{H}$ (resp. $\mathcal{A}$) is solvable.

Corollary 4.3 *Let $(\mathfrak{g}, B)$ be a solvable quadratic Lie superalgebra but neither $\{0\}$ nor the one-dimensional Lie algebra. Then, $(\mathfrak{g}, B)$ is obtained by a sequence of double extensions of solvable quadratic Lie superalgebras by the one-dimensional Lie algebra and/or generalized double extensions of solvable quadratic Lie superalgebras by the one-dimensional Lie superalgebra and/or orthogonal direct sums of solvable quadratic Lie superalgebras constructed in that way.*

5 Every Solvable Quadratic Lie Superalgebra is described by a T^*-extension

The following central Theorem was communicated to us by M. Duflo. We have made its proof more elementary upon circumventing the use of universal envelopping algebra which was used in Duflo's original proof.

Theorem 5.1 (Duflo) *Let $\mathfrak{g}$ be a solvable Lie superalgebra and $\mathcal{V}$ be a finite-dimensional $\mathfrak{g}$-module. If $\mathcal{V}$ is simple, non-zero and isomorphic to its dual $\mathcal{V}^*$, then $\mathcal{V}$ is the one-dimensional trivial $\mathfrak{g}$-module.*

Proof. Let $\mathcal{V}$ be a simple module of $\mathfrak{g}$. We shall denote by ρ the module morphism $\mathfrak{g} \to \mathrm{Hom}_{\mathbb{K}}(V,V)$, and by ρ^* the contragredient module map.
1. We shall first deal with the case that $\dim\mathcal{V} = 1$. It follows that $\mathcal{V} = \mathbb{K}e$. Let $\{f\}$ be the dual basis vector of $\{e\}$. Then there is a 1-form $\lambda \in \mathfrak{g}^*$ such that for each homogeneous element X of $\mathfrak{g}$ we have $\rho(X)e = \lambda(X)e$ and $\rho^*(X)f = -(-1)^{|X||e|}\lambda(X)f$. Let $L : \mathcal{V}^* \to \mathcal{V}$ be an isomorphism of $\mathfrak{g}$-modules. Then for each homogeneous element X of $\mathfrak{g}$ the fact that $L(\rho^*(X)f) = \rho(X)(L(f))$ implies that $(1 + (-1)^{|X||e|})\lambda(X) = 0$. Consequently, $\lambda(X) = 0$ if $|X| = 0$. Since $\rho(X)e = \lambda(X)e$, it follows that $\lambda(X) = 0$ if $|X| = 1$ because the linear map $\rho(X)$ changes parity and e is homogeneous. We conclude that $\mathcal{V}$ is a trivial $\mathfrak{g}$-module.
We are going to prove this Theorem now by induction on the dimension of $\mathfrak{g}$. We can and shall suppose henceforth that $\dim\mathcal{V} \geq 2$.
2.1 If $\dim\mathfrak{g} = 0$, then $\mathcal{V}$ is a simple $\mathbb{K}$-vector space, whence $\dim\mathcal{V} = 1$ which contradicts the above hypothesis.
2.2 If $\dim\mathfrak{g} = 1$, then $\mathfrak{g} = \mathbb{K}X$ where X is a homogeneous element of $\mathfrak{g}$. Since $\mathbb{K}$ is algebraically closed, it follows that $\rho(X)$ has a nonzero eigenvector $e \in \mathcal{V}\backslash\{0\}$ such that $\rho(X)e = \lambda e$ and $\lambda \in \mathbb{K}$. Let $e = e_{\overline{0}} + e_{\overline{1}}$ with $e_{\overline{0}} \in \mathcal{V}_{\overline{0}}$ and $e_{\overline{1}} \in \mathcal{V}_{\overline{1}}$. In case X is even it follows that both $e_{\overline{0}}$ and $e_{\overline{1}}$ are eigenvectors of X for the eigenvalue λ. Since at least one of them is nonzero, there would exist a proper submodule of $\mathcal{V}$ of dimension 1 which contradicts $\dim\mathcal{V} \geq 2$. In case X is odd, we have $\rho(X)(e_{\overline{0}}) = \lambda e_{\overline{1}}$ and $\rho(X)(e_{\overline{1}}) = \lambda e_{\overline{0}}$. Therefore the subspace generated by $e_{\overline{0}}$ and $e_{\overline{1}}$ is a submodule of $\mathcal{V}$, hence equal to it by simplicity of $\mathcal{V}$. If $e_{\overline{0}}$ or $e_{\overline{1}}$ was zero, there would again be a onedimensional submodule, contradiction. It follows that $e_{\bar{0}} \neq 0$ and $e_{\bar{1}} \neq 0$, whence V is 2-dimensional and spanned by $e_{\overline{0}}$ and $e_{\overline{1}}$. Consider the dual basis $\{f_{\bar{0}}, f_{\bar{1}}\}$ of $\mathcal{V}^*$ associated to $\{e_{\bar{0}}, e_{\bar{1}}\}$. It follows that $\rho^*(X)f_{\bar{0}} = -\lambda f_{\bar{1}}$ and $\rho^*(X)f_{\bar{1}} = \lambda f_{\bar{0}}$. Since $\mathcal{V}$ and its dual $\mathcal{V}^*$ are isomorphic $\mathfrak{g}$-modules, we can consider an isomorphism of $\mathfrak{g}$-modules $L : \mathcal{V}^* \to \mathcal{V}$. Define matrix elements $L_{\overline{a}\ \overline{b}} \in \mathbb{K}$ of L by $L(f_{\overline{b}} = L_{\overline{0}\ \overline{b}}e_{\overline{0}} + L_{\overline{1}\ \overline{b}}e_{\overline{1}}$ for all $\overline{a},\ \overline{b}\ \in \mathbb{Z}_2$. A little computation shows that the isomorphism condition $L \circ \rho^*(X) = \rho(X) \circ L$ implies that $\lambda = 0$. But then for instance $\mathbb{K}e_{\overline{0}}$ would be one-dimensional submodule, contradiction.
2.3 Suppose now that $\dim\mathfrak{g} \geq 2$, and the Theorem is true for all solvable Lie superalgebras $\mathcal{H}$ such that $\dim\mathcal{H} < \dim\mathfrak{g}$. The fact that $\mathfrak{g}$ is solvable implies that there exists a graded ideal $\mathfrak{h}$ of $\mathfrak{g}$ of codimension 1 such that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{h} \subset \mathfrak{g}$. There also exists a homogeneous element $T \in \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{K}T$ (direct sum of vector spaces). Moreover, let $\mathcal{W} \subset \mathcal{V}$ be a simple $\mathfrak{h}$-submodule of $\mathcal{V}$. Clearly, $\mathcal{W}$ is not equal to $\mathcal{V}$ since otherwise $\mathcal{W} = \mathcal{V}$ would be one-dimensional thanks to the induction hypothesis applied to $\mathfrak{h}$: this would contradict the assumption $\dim\mathcal{V} \geq 2$. Finally, for any nonnegative integer i let $\mathcal{W}_i$ denote the following vector subspace of $\mathcal{V}$:

$$\mathcal{W}_i := \mathcal{W} + \rho(T)\mathcal{W} + \cdots + \rho(T)^i\mathcal{W}. \tag{25}$$

For strictly negative i we set $\mathcal{W}_i := \{0\}$. We are going to show the following statements:

$$\text{There is a strictly positive integer } M \text{ such that } \rho(T)\mathcal{W}_M \subset \mathcal{W}_M \text{ and } \rho(T)\mathcal{W}_{M-1} \not\subset \mathcal{W}_{M-1}; \text{moreover if } |T| = \overline{1} \text{ then } M = 1. \quad (26)$$

$$\mathcal{W}_i \text{ is a } \mathfrak{h}\text{-module for all } i \in \mathbb{N}, \text{and } \mathcal{W}_M = \mathcal{V}. \quad (27)$$

$$\text{The } \mathfrak{h}\text{-module } \mathcal{W}_i/\mathcal{W}_{i-1} \text{ is isomorphic to } \mathcal{W} \text{ for all integers } 0 \leq i \leq M \quad (28)$$

Indeed, to show statement (26), it is clear that $\rho(T)\mathcal{W}_i \subset \mathcal{W}_{i+1}$, and since $\mathcal{V}$ is finite-dimensional there is a nonnegative integer N such that $\mathbf{1}_{\mathcal{V}}, \rho(T), \ldots, \rho(T)^{N+1}$ are linearly dependent which implies that $\rho(T)\mathcal{W}_N \subset \mathcal{W}_N$. Then M will be the minimum of all these integers. The case $M = 0$ would imply that $\mathcal{W}_0 = \mathcal{W}$ is a $\mathfrak{g}$-module contradicting the simplicity of the $\mathfrak{g}$-module $\mathcal{V}$ whence $M \geq 1$. Note that for odd T we have $\rho([T,T]) = 2\rho(T)^2$ with $[T,T] \in [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{h}$ showing $M = 1$ since $\mathcal{W}$ is a $\mathfrak{h}$-module.
Statement (27) follows for odd T immediately from the representation identity (for all homogeneous $x \in \mathfrak{g}$)

$$\rho(x)\rho(T) = \rho([x,T]) + (-1)^{|T||x|}\rho(T)\rho(x) \quad (29)$$

since $M = 1$ in that case, and $W_i = W_M$ for all integers $i \geq M$. For even T we use the following identity

$$\rho(x)\rho(T)^i = \sum_{k=0}^{i} \binom{i}{k} \rho(T)^k \rho(\underbrace{[\ldots[x,T]\ldots,T]}_{i-k \text{ brackets}}) \quad \forall\, x \in \mathfrak{g} \text{ and } \forall\, i \in \mathbb{N}, \quad (30)$$

(deduced from the representation identity (29) by induction) for the particular case $x \in \mathfrak{h}$ to prove the statement since the iterated brackets $[\ldots[x,T]\ldots,T]$ are then contained in the ideal $\mathfrak{h}$. Since the $\mathfrak{h}$-module $\mathcal{W}_M$ is also stable by $\rho(T)$ according to (26) it is a nonzero $\mathfrak{g}$-submodule of the simple $\mathfrak{g}$-module $\mathcal{V}$ and therefore equal to $\mathcal{V}$.
For statement (28) consider the linear map

$$\Phi_i : \mathcal{W} \to \mathcal{W}_i/\mathcal{W}_{i-1} : w \mapsto (-1)^{|T||w|}\rho(T)^i(w) + \mathcal{W}_{i-1}.$$

for homogeneous $w \in \mathcal{W}$ and all integers $0 \leq i \leq M$. Since $\mathcal{W}_i = \rho(T)^i\mathcal{W} + \mathcal{W}_{i-1}$ it follows that Φ_i is surjective. For odd T it suffices to use the representation identity (29) to show that Φ_i is a morphism of $\mathfrak{h}$-modules in view of the fact that $M = 1$, whereas for even T we use identity (30) to prove that each Φ_i is a morphism of $\mathfrak{h}$-modules. In both cases the kernel of Φ_i is an $\mathfrak{h}$-submodule of the simple $\mathfrak{h}$-module $\mathcal{W}$ so it is either equal to $\mathcal{W}$ or equal to $\{0\}$: in the first case Φ_i would be the zero map implying $\mathcal{W}_i = \mathcal{W}_{i-1}$ which is a contradiction to the definition of the integer M. It follows that Φ_i is injective and hence an isomorphism of $\mathfrak{h}$-modules.

We continue the proof of the Theorem: statement (28) implies in particular for $i = M$ that the $\mathfrak{h}$-module $\mathcal{W}$ is isomorphic to the quotient module $\mathcal{V}/\mathcal{W}_{M-1}$. Hence the dual $\mathfrak{h}$-module $\mathcal{W}^*$ is isomorphic to the $\mathfrak{h}$-module $(\mathcal{V}/\mathcal{W}_{M-1})^*$ which in turn is canonically isomorphic the the $\mathfrak{h}$-submodule

$$\mathcal{W}_{M-1}^{\mathrm{ann}} := \{f \in V^* \mid f(v) = 0 \;\forall\, v \in \mathcal{W}_{M-1}\} \subset \mathcal{V}^*$$

by means of the map $\psi : \mathcal{W}_{M-1}^{\mathrm{ann}} \to (\mathcal{V}/\mathcal{W}_{M-1})^*$ defined by $\psi(f)(v + \mathcal{W}_{M-1}) := f(v)$ (for all $f \in \mathcal{W}_{M-1}^{\mathrm{ann}}$ and $v \in \mathcal{V}$). Consider now an isomorphism of $\mathfrak{g}$-modules $L : \mathcal{V}^* \to \mathcal{V}$, and let $\tilde{\mathcal{W}}$ be the $\mathfrak{h}$-submodule of $\mathcal{V}$ defined by $\tilde{\mathcal{W}} := L(\mathcal{W}_{M-1}^{\mathrm{ann}})$. Since for each nonnegative integer i we have $\mathcal{W}_i \subset \mathcal{W}_{i+1}$ it follows that there is a smallest integer k such that $\tilde{\mathcal{W}} \subset \mathcal{W}_k$ and $\tilde{\mathcal{W}} \not\subset \mathcal{W}_{k-1}$.

The image of $\tilde{\mathcal{W}}$ under the $\mathfrak{h}$-module map $L' : \mathcal{W}_k \to \mathcal{W}_k/\mathcal{W}_{k-1} \cong W$ is an $\mathfrak{h}$-submodule of $\mathcal{W}$, hence it is either equal to $\{0\}$ or to $\mathcal{W}$. If the image was equal to $\{0\}$, then $\tilde{\mathcal{W}} \subset \mathcal{W}_{k-1}$, a contradiction, and hence the image is equal to $\mathcal{W}$. Since

$$\dim \tilde{\mathcal{W}} = \dim(\mathcal{W}^{\mathrm{ann}}_{M-1}) = \dim W^* = \dim W$$

it follows that $\tilde{\mathcal{W}} \cap \mathcal{W}_{k-1} = \{0\}$ whence the composition of $L' \circ L$ restricted to $\mathcal{W}^{\mathrm{ann}}_{M-1}$ gives an isomorphism of the $\mathfrak{h}$-module W^* with the $\mathfrak{h}$-module W. By the induction hypothesis applied to $\mathfrak{h}$ it follows that $\mathcal{W}$ is a one-dimensional trivial $\mathfrak{h}$-module. Again the representation identity (29) in case T is odd, and the identity (30) for even T show that then $\mathcal{W}_M = \mathcal{V}$ is a trivial $\mathfrak{h}$-module. Therefore $\mathcal{V}$ can in a canonical way be seen as a simple module of the one-dimensional quotient algebra $\mathfrak{g}/\mathfrak{h}$ satisfying the hypothesis of the Theorem. But this case has already been ruled out in **2.2**. Hence the hypothesis $\dim \mathcal{V} \geq 2$ always leads to a contradiction, and this proves the Theorem.

Remark 5 The proof of Theorem 5.1 shows that this Theorem is still true in the case when $\mathbb{K}$ is algebraically closed with characteristic $p \neq 2$.

Corollary 5.2 *Let $\mathfrak{g}$ be a solvable Lie superalgebra and $\mathcal{V}$ be a non-zero finite-dimensional $\mathfrak{g}$-module which admits a non-degenerate, even, supersymmetric and $\mathfrak{g}$-invariant bilinear form B. If* $\dim\mathcal{V} \geq 2$, *then there exists a non-zero isotropic $\mathfrak{g}$-submodule $\mathcal{W}$ of $\mathcal{V}$.*

Proof. If $\dim\mathcal{V} \geq 2$, then, by Theorem 5.1, $\mathcal{V}$ is a non-simple $\mathfrak{g}$-module. Consequently there exists a simple $\mathfrak{g}$-submodule $\mathcal{M}$ of $\mathcal{V}$, in particular $\mathcal{M} \neq \{0\}$, and $\mathcal{M} \neq \mathcal{V}$. Since $\mathcal{M}^{\perp} \cap \mathcal{M}$ is a $\mathfrak{g}$-submodule of $\mathcal{M}$, then $\mathcal{M}^{\perp} \cap \mathcal{M} = \{0\}$ or $\mathcal{M}^{\perp} \cap \mathcal{M} = \mathcal{M}$. It follows that either $B_{|\mathcal{M}\times\mathcal{M}}$ is non-degenerate or $\mathcal{M} \subseteq \mathcal{M}^{\perp}$.

If $\mathcal{M} \subseteq \mathcal{M}^{\perp}$, then $\mathcal{M}$ is a non-zero isotropic $\mathfrak{g}$-submodule of $\mathcal{V}$.

If $B_{|\mathcal{M}\times\mathcal{M}}$ is non-degenerate, then, by Theorem 5.1, $\dim\mathcal{M} = 1$ and $\mathcal{M}$ is a trivial $\mathfrak{g}$-module. Moreover, $\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Consider now a simple $\mathfrak{g}$-submodule $\mathcal{M}'$ of $\mathcal{M}^{\perp}$. If $\mathcal{M}'$ is isotropic, then the proof is finished. If $\mathcal{M}'$ is not isotropic, then $B_{|\mathcal{M}'\times\mathcal{M}'}$ is non-degenerate. It follows by Theorem 5.1, that $\mathcal{M}'$ is the one-dimensional trivial $\mathfrak{g}$-module. Consequently, the $\mathfrak{g}$-module $\mathcal{M} \oplus \mathcal{M}'$ is a 2-dimensional trivial nondegenerate $\mathfrak{g}$-module, and we can choose an isotropic one-dimensional graded vector sub-space $\mathcal{W}$ of $\mathcal{M} \oplus \mathcal{M}'$. Therefore $\mathcal{W}$ is a non-zero isotropic $\mathfrak{g}$-submodule of $\mathcal{V}$.

The following Lemma is the Lie superalgebra analogue of Lemma 3.2 of [6]:

Lemma 5.3 *Let $\mathcal{V}$ be a non-zero finite dimensional $\mathbb{Z}_2$-graded vector space which admits a non-degenerate, even and supersymmetric bilinear form B. Let $\mathcal{L}$ a solvable Lie sub-superalgebra of $\mathfrak{osp}(\mathcal{V}, B)$ and $\mathcal{W}$ a graded vector subspace of $\mathcal{V}$.*

If $\mathcal{W}$ is an isotropic $\mathcal{L}$-submodule of $\mathcal{V}$, then there exists a graded vector subspace $\mathcal{W}_{max}$ of $\mathcal{V}$ such that:

(i) $\mathcal{W} \subseteq \mathcal{W}_{max}$;

(ii) $\mathcal{W}_{max}$ *is an isotropic $\mathcal{L}$-submodule of $\mathcal{V}$;*

(iii) $\mathcal{W}_{max}$ *is maximal among all isotropic graded vector sub-space of $\mathcal{V}$;*

(iv) $\dim\mathcal{W}_{max} = [\frac{n}{2}]$ *(i.e. the integer part of* $\frac{n}{2}$*);*

(v) If n *is even,* $\mathcal{W}_{max} = (\mathcal{W}_{max})^{\perp}$*;*

(vi) If n *is odd,* $\mathcal{W}_{max} \subseteq (\mathcal{W}_{max})^{\perp}$*,* $\dim(\mathcal{W}_{max})^{\perp} - \dim\mathcal{W}_{max} = 1$*, and* $f((\mathcal{W}_{max})^{\perp}) \subseteq \mathcal{W}_{max}$*, for all* $f \in \mathcal{L}$*.*

Proof. We will prove this Lemma by induction on the dimension of $\mathcal{V}$. It is clear that that the lemma is true if $\dim\mathcal{V} = 1$.

Now, let us assume that the lemma is true if $\dim\mathcal{V} < n$, and we shall show it when $\dim\mathcal{V} = n$, where $n \geq 2$. By the Corollary 5.2, there exists $\mathcal{W}$ a non-zero isotropic $\mathcal{L}$-submodule of $\mathcal{V}$. If $\mathcal{W} = \mathcal{W}^{\perp}$, then $\dim\mathcal{W} = \frac{n}{2}$, it follows that $\mathcal{W}$ is maximal among all isotropic graded vector subspaces of $\mathcal{V}$. Consequently, $\mathcal{W}_{max} = \mathcal{W}$. If now $\mathcal{W} \neq \mathcal{W}^{\perp}$, we consider $\mathcal{V}' = \mathcal{W}^{\perp}/\mathcal{W}$ which is a $\mathbb{Z}_2$-graded vector space and $\dim\mathcal{V}' < \dim\mathcal{V} = n$ Moreover, the well defined map $\bar{B} : \mathcal{V}' \times \mathcal{V}' \to \mathbb{K}$ defined by $\bar{B}(X + \mathcal{W}, Y + \mathcal{W}) := B(X,Y), \forall X, Y \in \mathcal{W}^{\perp}$, is a non-degenerate, even and supersymmetric bilinear form. Now, if $f \in \mathcal{L}$, then the linear map $\bar{f} : \mathcal{V}' \to \mathcal{V}'$ defined by $\bar{f}(X + \mathcal{W}) := f(X) + \mathcal{W}, \forall X \in \mathcal{W}^{\perp}$, is well-defined. It is clear that $\mathcal{L}' = \{\bar{f} : f \in \mathcal{L}\}$ is a solvable Lie sub-superalgebra of $\mathfrak{osp}(\mathcal{V}', \bar{B})$. It follows, by the induction hypothesis, that there exists a graded vector subspace $\mathcal{W}' = \{0\}_{max}$ of $\mathcal{V}'$ which verifies the six assertions of the lemma. Let us consider $\mathcal{W}_{max} := S^{-1}(\mathcal{W}')$, where $S : \mathcal{W}^{\perp} \to \mathcal{W}^{\perp}/\mathcal{W}$ is the canonical surjection. Then, $\mathcal{W}_{max}$ is a graded vector subspace of $\mathcal{V}$ such that $\mathcal{W} \subseteq \mathcal{W}_{max}$ and $\mathcal{W}'$ is isomorphic to $\mathcal{W}_{max}/\mathcal{W}$. Consequently, $\dim\mathcal{W}_{max} = [\frac{n}{2}]$.

Let $X, Y \in \mathcal{W}_{max}$. Since $S(X)$ and $S(Y)$ are elements of $\mathcal{W}'$, then $B(X,Y) = \bar{B}(S(X), S(Y)) = 0$. This proves that $\mathcal{W}_{max}$ is isotropic. Consequently $\mathcal{W}_{max}$ is maximal among all isotropic graded vector subspaces of $\mathcal{V}$ because $\dim\mathcal{W}_{max} = [\frac{n}{2}]$. Let $X \in \mathcal{W}_{max}$, $f \in \mathcal{L}$, we have $S(f(X)) = \bar{f}(S(X)) \in \mathcal{W}'$ then $f(X) \in \mathcal{W}_{max}$. It follows that $\mathcal{W}_{max}$ is a $\mathcal{L}$-submodule of $\mathcal{V}$.

If n is even it is clear that $\dim\mathcal{W}_{max} = \dim(\mathcal{W}_{max})^{\perp} = \frac{n}{2}$. Consequently, $\mathcal{W}_{max} = (\mathcal{W}_{max})^{\perp}$.

If n is odd, then there exists $k \in \mathbb{N}$ such that $n = 2k + 1$ and $\dim\mathcal{W}_{max} = k$. Consequently $\dim(\mathcal{W}_{max})^{\perp} - \dim\mathcal{W}_{max} = 1$. Now, $\mathcal{W}_{max}^{\perp}/\mathcal{W}_{max}$ is an $\mathcal{L}$-module and the map $\bar{B} : \mathcal{W}_{max}^{\perp}/\mathcal{W}_{max} \times \mathcal{W}_{max}^{\perp}/\mathcal{W}_{max} \to \mathbb{K}$ defined by $\bar{B}(X + \mathcal{W}_{max}, Y + \mathcal{W}_{max}) := B(X,Y), \forall X, Y \in (\mathcal{W}_{max})^{\perp}$, is a $\mathcal{L}-$invariant non-degenerate, even and supersymmetric bilinear form. Then, by Theorem 5.1, $\mathcal{W}_{max}^{\perp}/\mathcal{W}_{max}$ is a trivial $\mathcal{L}$-module and, therefore, $f((\mathcal{W}_{max})^{\perp}) \subseteq \mathcal{W}_{max}$, for all $f \in \mathcal{L}$.

Theorem 5.4 *Let* $(\mathfrak{g}, B)$ *be a solvable quadratic Lie superalgebra. Then* $\mathfrak{g}$ *contains an isotropic graded ideal* $\mathfrak{I}$ *of dimension* $[\frac{\dim\mathfrak{g}}{2}]$ *which is maximal among all isotropic graded vector subspaces of* $\mathfrak{g}$*. Moreover, if* $\dim\mathfrak{g}$ *is even then* $(\mathfrak{g}, B)$ *is isometric to some* T^*-*extension of the Lie superalgebra* $\mathfrak{g}/\mathfrak{I}$*. If* $\dim\mathfrak{g}$ *is odd then* $(\mathfrak{g}, B)$ *is isometric to a non-degenerate graded ideal of codimension one in some* T^*-*extension of the Lie superalgebra* $\mathfrak{g}/\mathfrak{I}$*.*

Proof. The theorem is true if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$ (i.e. $\mathfrak{g}$ is abelian). Now, suppose that $\mathfrak{g}$ is not abelian.

Suppose que $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$. Since $\mathfrak{g}$ is solvable, then $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ and $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ because $[\mathfrak{g}, \mathfrak{g}]^{\perp} = \mathfrak{z}(\mathfrak{g})$. If we suppose that the graded ideal $[\mathfrak{g}, \mathfrak{g}]$ is non-degenerate then $\mathfrak{z}([\mathfrak{g}, \mathfrak{g}]) \neq \{0\}$ which contradicts the fact that $\mathfrak{z}([\mathfrak{g}, \mathfrak{g}]) \subseteq [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$. Consequently, $[\mathfrak{g}, \mathfrak{g}]$ is degenerate. Therefore, $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$. By Lemma 5.3, there exists an isotropic graded ideal $\mathfrak{I} := (\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}])_{max}$ of $\mathfrak{g}$ such that $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{I}$, $\mathfrak{I}$ is maximal among all isotropic graded vector subspaces of $\mathfrak{g}$ and $\dim\mathfrak{I} = [\frac{\dim\mathfrak{g}}{2}]$. If the dimension of $\mathfrak{g}$ is even, then $\mathfrak{g}$ is isometric

to a T^*-extension of $\mathfrak{g}/\mathfrak{I}$. Now, if the dimension of $\mathfrak{g}$ is odd, it follows, by Lemma 5.3, that $\dim\mathfrak{I}^\perp - \dim\mathfrak{I} = 1$ and $[\mathfrak{g}, \mathfrak{I}^\perp] \subseteq \mathfrak{I}$. Since B is non-degenerate and invariant, then $[\mathfrak{I}^\perp, \mathfrak{I}^\perp] = \{0\}$. Since $\dim\mathfrak{g}_{\bar{1}}$ is even, then $\dim\mathfrak{I}_{\bar{1}} = \frac{\dim\mathfrak{g}_{\bar{1}}}{2}$, $\dim\mathfrak{I}_{\bar{0}} = [\frac{\dim\mathfrak{g}_{\bar{0}}}{2}]$ and $\dim(\mathfrak{I}_{\bar{0}})^\perp \cap \mathfrak{g}_{\bar{0}} - \dim\mathfrak{I}_{\bar{0}} = 1$. It follows that $(\mathfrak{I}_{\bar{0}})^\perp \cap \mathfrak{g}_{\bar{0}}$ is not an isotropic vector subspace of $\mathfrak{g}_{\bar{0}}$. Consequently, there exists $x \in (\mathfrak{I}_{\bar{0}})^\perp \cap \mathfrak{g}_{\bar{0}}$ such that $B(x, x) \neq 0$. The fact that $\mathbb{K}$ is algebraically closed implies that there exists $\alpha \in \mathbb{K}$ such that $B(\alpha x, \alpha x) = -1$.

Let us consider the one-dimensional Lie algebra $\mathcal{E} = \mathbb{K}e$ with its invariant scalar product q defined by $q(e, e) := 1$, and the orthogonal direct sum $(\mathcal{A} = \mathfrak{g} \oplus \mathcal{E}, T := B \perp q)$ which is a quadratic Lie superalgebra such that $\mathcal{A}_{\bar{0}} = \mathfrak{g}_{\bar{0}} \oplus \mathcal{E}$ and $\mathcal{A}_{\bar{1}} = \mathfrak{g}_{\bar{1}}$. Next we consider $f := e + \alpha X \in \mathcal{A}_{\bar{0}}$ and $\mathcal{H} = \mathfrak{I} \oplus \mathbb{K}f$. It is obvious that $\mathcal{H}$ is an isotropic graded ideal of $\mathcal{A}$ such that $\dim\mathcal{H} = \frac{\dim\mathcal{A}}{2}$. It follows that $\mathcal{A}$ is isometric to a T^*-extension of the Lie superalgebra $\mathcal{A}/\mathcal{H}$. Let $\Phi : \mathcal{A} \to \mathfrak{g}/\mathfrak{I}$ be the linear map defined by: $\Phi(x + \lambda e) := (x - \lambda\alpha X) + \mathfrak{I}$, $\forall(x, \lambda) \in \mathfrak{g} \times \mathbb{K}$. It is a surjective morphism of Lie superalgebras such that $\mathrm{Ker}\Phi = \mathcal{H}$. Consequently, the Lie superalgebras $\mathfrak{g}/\mathfrak{I}$ and $\mathcal{A}/\mathcal{H}$ are isomorphic. We conclude that $\mathcal{A}$ is isometric to a T^*-extension of the Lie superalgebra $\mathfrak{g}/\mathfrak{I}$, and $\mathfrak{g}$ is isometric to a non-degenerate graded ideal $\mathfrak{g} \oplus \{0\}$ of codimension 1 of $\mathcal{A}$.

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